A study of Σ^2 singularities in the 3-R$\underline{\text{P}}$S Parallel Manipulator

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Abstract

This paper presents some analytical results related to the determination of the singular poses of the 3-R$\underline{\text{P}}$S parallel manipulator at which it gains two degrees of freedom. The forward kinematic univariate (FKU) of the manipulator acquires a special structure at such a pose. All such poses have been identified in the closed-form, using a Stüdy-parameter representation of SE(3), for both the operation modes of the 3-R$\underline{\text{P}}$S. These results are novel, to the best of the knowledge of the authors, and these have been verified using the traditional method, using the criterion of loss of rank of certain Jacobian matrices. The theoretical results have been illustrated with numerical examples.

Keywords: Σ^2 singularity, 3-R$\underline{\text{P}}$S manipulator, gain of degree-of-freedom, Stüdy parameterisation, spatial parallel manipulators.

1 Introduction

In this paper, the problem of *gain-type singularities* is revisited, with the goal of determining the poses in which the manipulator gains two DoF (termed hereafter as the Σ^2 singularity, following the Thom-Boardman classification of singularities [3]). Such poses have been mentioned in [4], albeit without an exhaustive coverage. The authors are not aware of any further work in this regard, except for [5], wherein one such pose has been analysed in detail.

The analysis performed in this paper builds upon the ideas presented in [6] and [7]: it adopts the *Stüdy-parameter* representation of SE(3), (see, e.g., [8] for the fundamentals in this regard) and the resulting identification of the operational modes from the former, while identifying the singular poses using the concept of FKU from the latter. Also, it uses some improvisations in eliminating variables from systems of polynomial equations, where more formal methods, such as the computation of the Gröbner bases (see, e.g., [9]), or even the *resultants* appear to be computationally intractable at

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Such a singularity occurs *inside* the workspace of the manipulator, at which the manipulator's end-effector *gains* one or more degree(s) of freedom (DoF) [1]. This is conceptually similar to the *singularity of the second kind*, and are associated with the loss of rank of certain Jacobian matrices [2].

According to the Thom-Boardman classification of singularities of a differential map f, a point x belongs to the class Σ^i if the kernel of $Df(x)$, (i.e., the differential of f at x) is of dimension i.

this point. Using these techniques, the FKU is derived for the *operation modes* $x_0 = 0$ and $x_3 = 0$ (see [10] for more details on the "modes"). The required form of the FKU for Σ^2 singularities is identified, and used to derive the conditions leading to such a singularity. Finally, after extensive symbolic computations, the condition for Σ^2 singularity is obtained in closed-form, in terms of the architecture, and input variables alone. These results have been verified by reverting back to the traditional Jacobian-based analysis – first order singularities require certain Jacobian matrices to lose rank by one, while for the Σ^2 case, the loss of rank is by two, (see, e.g., [1, 5]). The concepts presented in this paper are fairly generic, and can be applied to similar analysis of any other manipulator.

The rest of the paper is organised as follows: the concept of Σ^1, Σ^2 singularities are discussed in Section 2. Section 3 describes the geometry of the 3-R$\underline{\text{P}}$S manipulator, based on which its FKU is derived in Section 4. Section 5 presents the theoretical results on the Σ^2 singularities of the 3-R$\underline{\text{P}}$S manipulator, which are numerically exemplified in Section 6. Finally, Section 7 presents the conclusions of the paper.

2 Analysis of the gain-type singularities based on the FKU

Forward kinematic analysis of a spatial parallel manipulator typically involves the following stages: (a) parametrisation of the pose of the moving platform in terms of some chosen variables; (b) construction of the *loop-closure* constraint equations, relating these *unknown* variables to the known *input* variables; and (c) determination of evaluation of the system of equations. The choice of the unknown variables to represent the motion of the moving platform can be the *passive joint variables*, as in [7], or task-space parameters, such as the Stüdy parameters, as in [6]. The constraint equations are then formulated based on the geometric conditions that the motion of the moving platform must satisfy. Finally, following appropriate mathematical procedures, the set of kinematic equations are reduced to a single univariate equation, termed as the FKU. The FKU contains *all* the information about the manipulator's kinematics, and can be used to study its singularities, as described below.

2.1 Σ^1 singularities

The *necessary* and *sufficient* condition for Σ^1 singularity is that the FKU is of the form:

$$f(x) : (x-\alpha)^{2m}(x-\beta)^{2n}\dots h(x) = 0, \quad m, n \in \mathbb{Z}^+,\ \alpha, \beta \in \mathbb{R}. \tag{1}$$

Consider, e.g., $x = \alpha$. When $m = 1$, the implication is clear: a pair of forward kinematic branches meet at $x = \alpha$, and hence the necessity and sufficiency of the condition is obvious from the statement of the *inverse function theorem*, which dictates that $f'(\alpha) = 0$ as well.

Noting that the function f has all the architecture parameters, and input variables in it, the kinematic conditions for Σ^1 singularity can be written as $f(x, \boldsymbol{\gamma}) = 0$, $f'(x, \boldsymbol{\gamma}) = 0$ (with a slight abuse of notation), where $\boldsymbol{\gamma}$ represents the vector of *known variables*. Eliminating the sole remaining unknown, x, between these equations, one obtains the condition for Σ^1 singularity purely in terms of the architecture parameters and the input variables [7].

2.2 Σ^2 singularities

The *necessary* condition for Σ^2 singularity, i.e., singularities corresponding to the gain of two DoF requires that two or more distinct pairs of forward kinematic branches meet in a pairwise manner at, say, $x = \alpha$. This requires that in Eq. (1), *at least one* of the indices $m, n, \cdots > 1$.

It is interesting to note, however, that when $m > 1$, say, $m = 2$, the point $x = \alpha$ does not *necessarily* lie on a Σ^2 singularity. In general, the form of Eq. (1) is only necessary for Σ^m singularity, and *not* sufficient, when $m > 1$. The same is true for the other points, $x = \beta$ etc. This is so since the implicit function theorem does not generalise in the manner one might have expected – i.e., if $m > 1$, though m pairs of solutions meet at a point, the derivative does not necessarily develop a kernel of dimension m. While there are many examples of manipulators where this generalisation holds good (the 3-RP̲S being one of them) there are counter examples where that is not the case [11]. This leads to the conclusion that the sufficiency conditions for the Σ^2 singularities cannot be derived based on the analysis of the FKU alone in the case of a general manipulator, and that it requires additional information, either on the structure of the FKU, or the actual kernel dimension of $Df(x)$ computed otherwise. However, the following strategy can be used to identify *all* the Σ^2 singularities in *any* manipulator:

- Identify all the potential cases of Σ^2 singularities, using the necessary condition,
- Check all the real solutions to isolate the cases satisfying $\mathrm{kernel}(Df(x)) = 2$.

Application of the above theory to the 3-RP̲S manipulator is detailed in Section 5.

3 Kinematic formulation of the 3-RP̲S manipulator

This section describes the geometry of the 3-RP̲S manipulator, and also derives the kinematics constraints the manipulator is subjected to.

3.1 Geometry of the 3-RP̲S manipulator

The 3-RP̲S parallel manipulator consists of a fixed base and a moving platform, both in the shape of equilateral triangles with circumradius 'b' and 'a', respectively. Without any loss of generality, the base radius, b, is considered to be *unity*. The triangular platforms are connected by three identical RP̲S legs, (see Fig. 1), wherein the rotational joints are attached to the base and their axes are tangent to the base circumcircle. The spherical joints are attached to the moving platform. The leg lengths can be varied by actuating the prismatic joints, thereby permitting the control of the three DoF motion of the moving-platform. The *homogeneous coordi-*

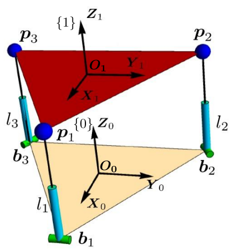

Figure 1: The 3-RP̲S parallel manipulator

nates of the vertices of the base platform, expressed in the base frame $\{0\}$, are: ${}^0\mathbf{b}_1 = \{b, 0, 0, 1\}^\top$, ${}^0\mathbf{b}_2 = \left\{-\frac{b}{2}, \frac{\sqrt{3}b}{2}, 0, 1\right\}^\top$, ${}^0\mathbf{b}_3 = \left\{-\frac{b}{2}, -\frac{\sqrt{3}b}{2}, 0, 1\right\}.^\top$ The coordinates of the vertices of the top platform are first found with reference to the moving frame $\{1\}$, and then transformed to the frame $\{0\}$ using the homogeneous transformation matrix ${}^0_1\boldsymbol{T}$ obtained in terms of the Stüdy parameters (see, e.g., [6]), $\boldsymbol{x} = \{x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3\}^\top$, i.e., ${}^0\boldsymbol{p}_i = {}^0_1\boldsymbol{T}\,{}^1\boldsymbol{p}_i,\ i = 1, 2, 3$.

3.2 Constraint equations

The 3-RP̲S manipulator has three active (i.e., input/known) variables, l_1, l_2, and l_3, representing the prismatic actuators in the legs, and eight passive *(i.e., output/unknown)* variables, x_i, y_i, $i = 0, \ldots, 3$, in the form of the Stüdy parameters. To obtain the eight unknown variables in terms of the known entities, eight *independent* equations are formulated as detailed below.

Planarity constraints: The vertex $\boldsymbol{p}_1$ is constrained to move in the $X_0 Z_0$ plane, as shown in Fig. 1, i.e., the Y_0 coordinate of the vector $({}^0\boldsymbol{p}_1 - {}^0\boldsymbol{b}_1)$ must be zero:

$$({}^0\boldsymbol{p}_1 - {}^0\boldsymbol{b}_1) \cdot \boldsymbol{e}_{Y_0} = 0, \text{ where } \quad \boldsymbol{e}_{Y_0} = \{0, 1, 0, 0\}^\top. \tag{2}$$

Similarly, the vertices $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$ are constrained to move in their respective planes:

$$\left\{\boldsymbol{T}_{Z_0}\left(\frac{-2\pi}{3}\right)({}^0\boldsymbol{p}_2 - {}^0\boldsymbol{b}_2)\right\} \cdot \boldsymbol{e}_{Y_0} = 0, \tag{3}$$

$$\left\{\boldsymbol{T}_{Z_0}\left(\frac{-4\pi}{3}\right)({}^0\boldsymbol{p}_3 - {}^0\boldsymbol{b}_3)\right\} \cdot \boldsymbol{e}_{Y_0} = 0, \tag{4}$$

where the homogeneous transformation matrix $\boldsymbol{T}_{Z_0}(\theta)$ corresponds to a CCW rotation about Z_0 by θ. Eqs. (2-4), after some manipulations, give rise to three planarity

These coordinates are obtained by a bijective transformation from the three dimensional affine space $\mathbb{R}^3$ to the projective space $\mathbb{P}^3$. In other words, these coordinates are obtained by mapping $\{x, y, z\}$ to $\{x, y, z, w\}$, with the projective coordinate $w \neq 0$ set to unity, without any loss of generality.

constraints, identical to those presented in [6]:

$$f_{A_1} : ax_1^2 - ax_2^2 - 2x_1y_0 + 2x_0y_1 - 2x_3y_2 + 2x_2y_3 = 0, \tag{5}$$

$$f_{A_2} : ax_1x_2 - 2ax_0x_3 + x_2y_0 - x_3y_1 - x_0y_2 + x_1y_3 = 0, \tag{6}$$

$$f_{A_3} : x_0x_3 = 0. \tag{7}$$

Leg length constraints: The respective distances between ${}^0\boldsymbol{p}_i$ and ${}^0\boldsymbol{b}_i$ yield three constraint equations, as shown below:

$$({}^0\boldsymbol{p}_i - {}^0\boldsymbol{b}_i) \cdot ({}^0\boldsymbol{p}_i - {}^0\boldsymbol{b}_i) = l_i^2, \quad i = 1, 2, 3. \tag{8}$$

The constraints due to the legs 1, 2, and 3 are formulated as $f_{A_4} = 0$, $f_{A_5} = 0$, and $f_{A_6} = 0$, respectively. The expressions are too big to be included here.
Stüdy quadric: In order to represent a rigid motion, the Stüdy parameters must lie on the *Stüdy quadric*:

$$f_{A_7} : x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0. \tag{9}$$

Normalisation constraint: This constraint ensures that the first four of the Stüdy paramters form a *unit quaternion*:

$$f_{A_8} : x_0^2 + x_1^2 + x_2^2 + x_3^2 - 1 = 0. \tag{10}$$

The set of eight constraint equations constitutes the *constraint ideal* $I_A = \langle f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}, f_{A_5}, f_{A_6}, f_{A_7}, f_{A_8} \rangle$. Following [6], *primary decomposition* of the polynomial ideal composed of the planarity constraints and Stüdy quadric alone, i.e., $\langle f_{A_1}, f_{A_2}, f_{A_3}, f_{A_7} \rangle$, is performed, to identify the operation modes of the 3-R$\underline{\text{P}}$S, characterised by $x_0 = 0$ and $x_3 = 0$, respectively.

4 Derivation of the FKU

In order to obtain the FKU, the passive variables are eliminated by adapting a sequential procedure for each operation mode, as described in the following.

4.1 Operation mode corresponding to $x_0 = 0, x_3 \neq 0$

In this mode, the elimination of the unknowns are done in the following steps:
1. Consider the ideal, I_A. Upon substituting $x_0 = 0$, the ideal I_A reduces to the ideal $J_A = \langle f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}, f_{A_5}, f_{A_6}, f_{A_7} \rangle$. The polynomial equations $f_{A_1} = 0, f_{A_2} = 0$ and $f_{A_6} = 0$ can be solved linearly for y_1, y_2 and y_3. Back-substitution of the expressions of y_1, y_2, and y_3 into J_A results in a new ideal $J_B = \langle f_{B_1}, f_{B_2}, f_{B_3}, f_{B_4} \rangle$ wherein, $f_{B_1} = f_{A_3}, f_{B_2} = f_{A_4}, f_{B_3} = f_{A_5}$ and $f_{B_4} = f_{A_7}$.
2. The polynomials f_{B_1} and f_{B_2} are both quadratic in y_0. Therefore, the remainder obtained by dividing f_{B_1} by f_{B_2}, while treating them as polynomials in y_0, can be solved linearly for y_0 as shown in Eq. (11).

$$y_0 = \frac{a(a+2)\left(3x_1^2 + 2\sqrt{3}x_2x_1 - 3x_2^2\right) + l_1^2 - l_2^2}{4a\left(3x_1 - \sqrt{3}x_2\right)}, \ a \neq 0. \tag{11}$$

In the following analysis, it is considered that the denominator in Eq. (11), i.e., $(3x_1 - \sqrt{3}x_2)$ is not zero, as it has been verified that vanishing of this factor finally leads only to a Σ^1 singularity. The denominator term a, being the circumradius of top platform, cannot be zero for a physically meaningful manipulator. Back-substituting y_0 into the ideal J_B, a new ideal $J_C = \langle f_{C_1}, f_{C_2}, f_{C_3}\rangle$ is formed, wherein, $f_{C_1} = f_{B_2}, f_{C_2} = f_{B_3}$, and $f_{C_3} = f_{B_4}$.
3. The polynomials f_{C_1} and f_{C_2} are of degree 8 in x_3. The remainder obtained by dividing f_{C_1} by f_{C_2} with respect to x_3, can be decomposed into 4 factors, given as:

$$\begin{aligned} f_{C_{1a}} &: 2\sqrt{3}x_2\left(x_1^2\left(-\left(9a(a+2)+l_1^2\right)\right)+x_2^2\left(3a(a+2)-l_1^2\right)-l_1^2x_3^2\right) \\ &\quad + l_2^2\left(3x_1+\sqrt{3}x_2\right)-l_3^2\left(3x_1-\sqrt{3}x_2\right), \end{aligned} \tag{12}$$

$$f_{C_{1b}} : \sqrt{3}(x_2-\sqrt{3}x_1), \qquad f_{C_{1c}} : x_3^2, \qquad f_{C_{1d}} : 4a^2. \tag{13}$$

It is already stated that $f_{C_{1c}}$ and $f_{C_{1d}}$ cannot be zero, and vanishing of the factor $f_{C_{1b}}$ leads to Σ^1 singularity. Therefore, only the factor $f_{C_{1a}}$ will be considered for further analysis, using the new ideal $J_D = \langle f_{D_1}, f_{D_2}, f_{D_3}\rangle$, where $f_{D_1} = f_{C_{1a}}, f_{D_2} = f_{C_2}$, and $f_{D_3} = f_{C_3}$.
4. The polynomials f_{D_1} and f_{D_2} are of degrees 2 and 8 in x_3, respectively. Upon dividing f_{D_2} by f_{D_1} with respect to x_3, the remainder obtained can be decomposed into 5 factors. These factors are denoted by $f_{D_{2a}}, \ldots, f_{D_{2e}}$. The factor $f_{D_{2a}}$ is of degree 4 in both x_1 and x_2. The expression of $f_{D_{2a}}$ is large, and hence is not reproduced here. The other factors are much smaller:

$$f_{D_{2b}} : 3\left(x_2-\sqrt{3}x_1\right)^2, \; f_{D_{2c}} : \left(3x_1^2-x_2^2\right)^2, \; f_{D_{2d}} : x_2^2, \; f_{D_{2e}} : 12a^4. \tag{14}$$

As before, vanishing of the factors $f_{D_{2b}}$, $f_{D_{2c}}$, and $f_{D_{2d}}$ lead to Σ^1 singularities, and a cannot be zero. Therefore, only $f_{D_{2a}}$ is considered as the member of the new ideal $J_E = \langle f_{E_1}, f_{E_2}, f_{E_3}\rangle$, where $f_{E_1} = f_{D_{2a}}, f_{E_2} = f_{D_1}, f_{E_3} = f_{D_3}$.
5. Both the polynomials f_{E_2} and f_{E_3} are of degree 2 in x_3. On dividing f_{E_2} by f_{E_3} with respect to x_3, the remainder $f_{E_{2a}}$ is of degree 2 in x_1 and degree 3 in x_2. The remainder is reproduced as:

$$f_{E_{2a}} : -2\sqrt{3}x_2\left(3a(a+2)\left(3x_1^2-x_2^2\right)+l_1^2\right)+l_2^2\left(3x_1+\sqrt{3}x_2\right)+l_3^2\left(\sqrt{3}x_2-3x_1\right).$$

On dividing the polynomials f_{E_1} by $f_{E_{2a}}$ with respect to x_1, the remainder obtained is a univariate polynomial in x_2, which decomposes into eight factors, denoted by $f_{E_{1a}}, \ldots, f_{E_{1h}}$. The factor $f_{E_{1a}}$ is large and of degree 8 in x_2, while $f_{E_{1h}}$ is the integer 432. The other factors are:

$$f_{E_{1b}} : x_2^4, \qquad f_{E_{1c}} : (l_1+l_3)^2, \qquad f_{E_{1d}} : (l_1-l_3)^2, \tag{15}$$

$$f_{E_{1e}} : (l_1+l_2)^2, \qquad f_{E_{1f}} : (l_1-l_2)^2, \qquad f_{E_{1g}} : (a+2)^4. \tag{16}$$

Conditions leading to only Σ^1 singularities are discarded, as the paper focusses on Σ^2 singularities.

The *size* of the expression $f_{D_{2a}}$ is 134.928 KB. Here, all the symbolic computations have been performed using the commercial computer algebra software, Mathematica and "size" refers to the amount of computer memory needed to store an expression in Mathematica's internal format.

Vanishing of the factors $f_{E_{1b}}, \ldots, f_{E_{1g}}$ lead to Σ^1 singularities. Therefore, $f_{E_{1a}}$ alone is analysed further. The polynomial $f_{E_{1a}}$ is quartic in x_2^2, is the desired FKU. It is cast in the following form:

$$FKU : c_0x^4 + c_1x^3 + c_2x^2 + c_3x + c_4, \qquad \text{where } x = x_2^2. \tag{17}$$

The coefficients c_i have been obtained in closed-form. However, owing to the constraint on space, only c_4 is quoted below:

$$\begin{aligned} c_4 = &- (r_2 - r_3)^4(9a^4 + 72a^3 - 6a^2(r_2 + r_3 - 30) - 24a(r_2 + r_3 - 6) \\ &+ (r_2 - r_3)^2 - 24(r_2 + r_3)), \qquad \text{where } r_i = l_i^2,\ i = 1, 2, 3. \end{aligned}$$

4.2 Operation mode corresponding to $x_3 = 0, x_0 \neq 0$

The derivation of the FKU follows the similar structure in this case, which is omitted here for the want of space. The final result is also similar:

$$\begin{aligned} &FKU : d_0x^4 + d_1x^3 + d_2x^2 + d_3x + d_4, \quad \text{where } x = x_1^2, \quad \text{and} \\ &d_4 = - (r_2 - r_3)^4(9a^4 - 72a^3 - 6a^2(r_2 + r_3 - 30) \\ &\qquad + 24a(r_2 + r_3 - 6) + (r_2 - r_3)^2 - 24(r_2 + r_3)). \end{aligned} \tag{18}$$

The other coefficients are too large to be presented here.

5 Σ^2 singularities of the 3-RP̲S manipulator

The kinematic conditions for Σ^2 singularities in the 3-RP̲S manipulator are derived in the following.

5.1 Structure of the FKU for Σ^2 singularities

The original FKU, as in Eq. (17) or Eq. (18), is converted into its monic form,

$$x^4 + t_1x^3 + t_2x^2 + t_3x + t_4 = 0, \text{ wherein } t_i = c_i/c_0, \text{ or } d_i/d_0, i = 1, \ldots, 4. \tag{19}$$

It is assumed above that $c_0 \neq 0, d_0 \neq 0$. In the most general case of Σ^1 singularity, Eq. (19) reduces to the form (see Eq. (1)):

$$(x - \alpha)^2(x - \beta)^2 = 0, \quad x = x_2^2, \text{or } x = x_1^2. \tag{20}$$

Comparing the equal powers of x between Eq. (19) and Eq. (20) one gets four equations in α, β. After eliminating α, β from these, the condition for Σ^1 singularities reduce to:

$$(4t_2 - t_1^2)^2 - 64t_4 = 0, \tag{21}$$

$$t_3^2 - t_1^2t_4 = 0. \tag{22}$$

The Σ^2 singularities occur only when more than one pair of forward kinematic branches meet at a point. It is obvious from Eq. (20) that the (necessary) condition for the same is simply: $\alpha = \beta$. Under this condition, Eq. (20) reduces to the form $(x - \alpha)^4 = 0$. In the following, the poses satisfying the condition $\alpha = \beta$ are identified analytically.

5.2 Identification of the Σ^2 singularities

For both the operation modes, i.e., $x_0 = 0$ and $x_3 = 0$, Eq. (21) is of degree 12 in r_1, degree 8 in both r_2 and r_3. Eq. (22) is of degree 6 in r_1, and degree 8 in both r_2 and r_3. In either case, the task is to eliminate one of the variables, r_i, to obtain a single equation in the other two, defining the Σ^2 singularity manifold. However, it becomes too complex to perform this last elimination step symbolically. Hence, for the mode $x_0 = 0$ the value $a = 3$ is adopted from [6]) and for the mode $x_3 = 0$ the value $a = 1/2$ is adopted from [5] for the following calculations. In $x_0 = 0$ mode, elimination of r_1 from Eq. (21) and Eq. (22) leads to an equation of degree 120 in r_2 and r_3 each, which decomposes into 9 factors, namely, f_i, $i = 1, \ldots, 9$. Only the factors $f_7 = 0$ and $f_8 = 0$ yields same structure as in Eq. (20) and lead to Σ^2 and Σ^1 singularities, respectively. The factor f_7 is shown below:

$$\begin{aligned} f_7 : {} & r_2^4 + 2r_3r_2^3 - 306r_2^3 + 3r_3^2r_2^2 - 576r_3r_2^2 + 33777r_2^2 \\ & + 2r_3^3r_2 - 576r_3^2r_2 + 51678r_3r_2 - 1586304r_2 + r_3^4 - 306r_3^3 \\ & + 33777r_3^2 - 1586304r_3 + 26873856. \end{aligned} \tag{23}$$

A similar set of results is obtained for the case $x_3 = 0, x_0 \neq 0$. Numerical examples of both the cases are given in Section 6 representing singular poses with a gain of two DoF.

6 Numerical results

In this section, numerical examples for both the operation modes are presented to demonstrate the validity of the theoretical results derived above.
Case 1: Mode $x_0 = 0$
For the design parameter $a = 3$, a combination of real values of r_2 and r_3 satisfying Eq. (23) are: $r_2 = 72$ and $r_3 = 9$, while a corresponding real solution of Eq. (22) is $r_1 = 72$. For these numbers, Eq. (17) becomes:

$$\begin{aligned} & 362.7970x^4 - 507.9158x^3 + 266.6558x^2 - 62.2196x + 5.4442 = 0 \\ \Rightarrow & (x - 0.35)^4 = 0, \quad \text{where } x = x_2^2. \end{aligned}$$

Therefore Σ^2 singularities occur at two distinct points, $x_2 = \pm\sqrt{0.3499}$. The corresponding singular poses are shown in Fig. 2. For the pose given by $x_2 = -0.5916$, (see Fig. 2(a)) the corresponding Stüdy parameters are: $\boldsymbol{x} = \{$0, -0.3415, -0.5916, -0.7302, -1.7078, -2.2135, 1.2780, $8.0540 \times 10^{-8}\}$. At this pose, the singular values of $\boldsymbol{J_{\eta\phi}}$ are $\{9071.9999, 0, 0\}$, and the singular values of $\boldsymbol{J_{\eta X}}$ (see Appendix A for the definitions of $\boldsymbol{J_{\eta\phi}}$ and $\boldsymbol{J_{\eta X}}$) are: $\{21735.7201, 1070.1392, 147.8653, 19.8730, 5.3520, 0.3835, 0, 0\}$.

The singularities can be classified based on the values of α and β, i.e., if they are real and distinct then the manipulator undergoes Σ^1 singularities, and when they are real and equal, the manipulator undergoes Σ^2 singularities.

The numbers quoted in this section are accurate only up to the fourth digit after the decimal point. However, in actual computations, 100 digits were retained. Also, any number less than 10^{-10} (in the absolute sense) is reported as zero.

(a) Pose A: $x_2 = -0.5916$

(b) Pose B: $x_2 = 0.5916$

Figure 2: Σ^2 singular poses for $x_0 = 0$

The two zero singular values confirm the Σ^2 nature of the singularity in either case.
Case 2: Mode $x_3 = 0$:
In this case, the chosen numerical values are: $a = 1/2$, $r_3 = 15/4$, and $r_2 = 9$, which satisfy condition for Σ^2 singularity. These values are then substituted in Eq. (22) and solved for r_1. Finally, on substituting $r_1 = 15/4$, $r_2 = 9$, and $r_3 = 15/4$, Eq. (18) becomes:

$$
\begin{aligned}
&20736x^4 - 48384x^3 + 42336x^2 - 16464x + 2401 = 0, \text{where } x = x_1^2;\\
\Rightarrow&(x - 7/12)^4 = 0.
\end{aligned}
$$

In this case too, it is verified that there are two vanishing singular values of the constraint Jacobian matrices.

7 Conclusions

An analytical study of the *gain-type* singularities of the 3-R$\underline{\text{P}}$S manipulator has been presented, with special focus on the identification of the set of poses where the manipulator gains two DoF. Such singularities are termed as the Σ^2 singularities, and these form a subset of the Σ^1 singularties, which result in the gain of a single DoF. The analytical results obtained from the study of the FKU have been verified numerically, using the standard Jacobian-based approach.

Appendix

A Definition of the constraint Jacobian matrices

At a gain-type singularity, certain constraint Jacobian matrices lose rank. In this work, two such matrices have been used to validate the results obtained by the proposed new conditions based on the properties of the FKU. One such Jacobian matrix is derived from the constraint equations in the joint space [7]: $\boldsymbol{\eta}(\boldsymbol{\theta}, \boldsymbol{\phi}) = \mathbf{0}$, where $\boldsymbol{\theta}$ represents the active variables, and ϕ the passive joint variables. The required Jacobian matrix arising out of this equation is derived in [7] as: $\boldsymbol{J}_{\eta\phi} = \frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{\phi}}$. Similarly, writing the constraint equations in the task-space as: $\boldsymbol{f}(\boldsymbol{\theta}, \boldsymbol{X}) = \mathbf{0}$, one can derive the relevant Jacobian matrix in this case as: $\boldsymbol{J}_{\eta X} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{X}}$. The latter Jacobian is the same as the one used in [6], and the system of equations $\boldsymbol{f}(\boldsymbol{\theta}, \boldsymbol{X}) = \mathbf{0}$ define the same ideal as I_A in Section 3.

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