Data-Driven Kinematics: Unifying Synthesis of Planar Four-Bar Linkages via Motion Analysis

Anurag Purwar, Q. Jeffrey Ge

Abstract

This paper presents a novel data-driven approach for kinematic synthesis of planar four-bar linkages consisting of revolute (R) or prismatic (P) joints by extracting the geometric constraints of a motion. The approach unifies the often elusive type synthesis problem with dimensional synthesis and for a given motion determines the best combinations of R- and P-joints in a four-bar linkage and their dimensions by analyzing a given motion. The underlying formulation is based on concepts of planar quaternions, kinematic mapping, and data-fitting. By formulating the kinematic constraints of planar four-bar linkages in a unified form and then fitting the constraints with the given motion data in the image space of planar displacements, we obtain the best type and the dimensions of the linkages. The results will be demonstrated via *MotionGen*, an intuitive iOS and Android app that implements this approach and allows designers to synthesize linkages for the motion generation problem.

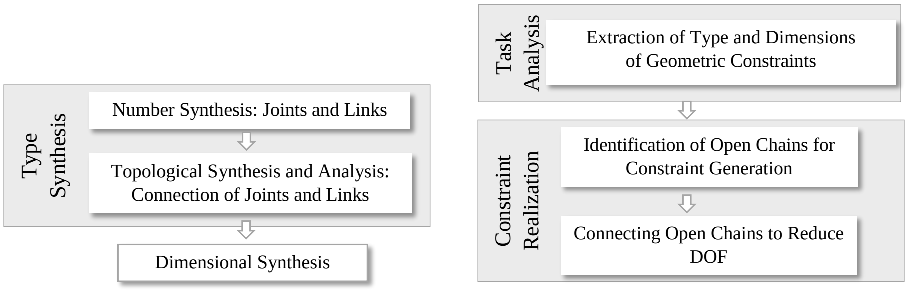
Keywords: Planar-Four Bar Linkage Synthesis and Analysis, Planar Quaternions, Kinematic Mapping, Data Fitting, Reverse Engineering

1 Introduction

The overarching goal of our current research is to bring together the Computational Shape Analysis and Design Kinematics to develop a new data-driven paradigm for kinematic synthesis of mechanical motion generating devices. We seek to establish a new computational foundation for simultaneous type and dimensional synthesis of such devices. This includes (1) the development of "freeform" or unified versions of design equations that span broad classes of mechanisms; (2) the development of unified algorithms for data-driven simultaneous type and dimensional synthesis of planar, spherical and spatial mechanisms. This effort to reformulate design kinematics to unify the type and dimensional synthesis could have a transformative effect on mechanism design in terms of both research and design practice. In this paper, our focus is on presenting unified synthesis approach for the planar four-bar motions, which leads to a simultaneous determination of both type and dimensions for a given motion. The algorithm developed is highly amenable to efficient implementation that permits real-time computation. This is further demonstrated via an iOS and Android *MotionGen* app developed at Stony Brook University. This intuitive app facilitates solving complex motion generation problems without requiring advanced knowledge of kinematics from the user.

Anurag Purwar (Corresponding author), Research Associate Professor
Computer-Aided Design Innovation Lab, Mechanical Engineering, Stony Brook University, Stony Brook, NY, anurag.purwar@stonybrook.edu .

Q. Jeffrey Ge, Professor and Interim Chair
Computational Design Kinematics Lab, Mechanical Engineering, Stony Brook University, Stony Brook, NY, qiaode.ge@stonybrook.edu



(a) Traditional approach (b) Data-driven approach

Figure 1: Traditional approach vs Data-driven approach: In the Data-driven approach, the first step is to extract the geometric constraints implicit in a given motion, and then to match them with appropriate mechanical dyads before connecting two of them to form a single-DOF mechanism.

1.1 Background

The current state of art in mechanism design is such that before carrying out the dimensional synthesis, one has to reach at a decision regarding the type of mechanism to be employed for a specified motion requirement. While dimensional synthesis has been a subject amenable to mathematical treatments, identifying a mechanism type that matches with a given motion task is often driven by the designer's past kinematic experience. This two-step paradigm for mechanism design has been around for over a century; see Fig. (1)(a). It is derived from the classical viewpoint that a mechanism is defined by links, joints and the pattern of their interconnections. While this has led to effective means for mechanism classification and enumeration [1, 2], it makes type synthesis a very challenging task, even for those who have been well trained in Mechanism Science. Erdman, Sandor, and Kota [3] have summarized the importance and challenges of the type synthesis problem. There have been attempts to solve the combined problem of type synthesis and dimensional synthesis through the use of Genetic Algorithm [4, 5], topology optimization [6, 7] as well as a uniform polynomial system [8]. However, they have been carried out for very restricted applications with very limited number of mechanism types. More significantly, these approaches do not reduce the complexity of the type synthesis problem. Researchers in AI (Artificial Intelligence) community have also sought with very limited success to bridge the gap between type and dimensional syntheses and developed what is known as *Qualitative Kinematics* [9] in the context of qualitative spatial reasoning.

The ability to decompose a design problem into type and dimensional synthesis is actually data dependent. For applications that require the coupling of rotations and translations such as the coupler motion in a single-DOF planar closed kinematic chain, the type synthesis problem may not be solved without engaging in dimensional synthesis simultaneously. This necessitates a data- or task-driven paradigm for simultaneous type and dimensional synthesis. The key is to establish a computational approach for transforming the problem of *selecting* a mechanism type into that which is directly *computable* from the specified task, i.e., the quantitative information about a specified task drives the mechanism design

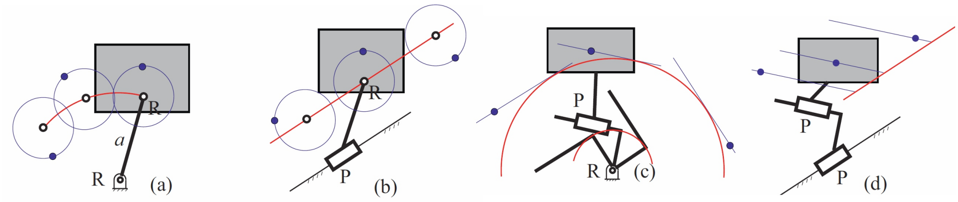


Figure 2: Planar dyads: RR, PR, RP, and PP

process, and is used to determine both the type and dimensions of a desired mechanism; see Fig. 1(b). In this paper, our focus is on demonstrating this data-driven approach for synthesis of planar four-bar linkages consisting of R- and P-joints.

2 Approach

The basic idea of data driven kinematic synthesis is to extract geometric constraints (both type and dimensions) from the input data (task positions), called *Task Analysis* in Fig. 1(b). In the following, we present data-driven synthesis of planar four-bar mechanisms to illustrate our approach.

2.1 Planar Dyads

Planar dyads with revolute (R) and prismatic (P) joints are the building blocks of planar four-bar linkages. A planar dyad defines a 2-DOF motion in the plane. Consider first RR and PR dyads shown in Figures 2(a) and 2(b). They impose point-based constraint such that its moving pivot lies on a circle (RR) or a line (PR). For RP and PP dyads shown in Figure 2(c) and 2(d), they impose line-based constraint such that the moving line in the P-joint stays tangent to a circle (RP) or parallel to another line (PP). By connecting two dyads, one obtains a 1-DOF planar four-bar linkage.

2.2 Data-Driven Fitting of Circles and Lines

Let (X, Y) denote coordinates of a point in the fixed plane. The following implicit equation

$$a_0(X^2 + Y^2) + 2a_1X + 2a_2Y + a_3 = 0, \tag{1}$$

represents a circle (when $a_0 \neq 0$) and a line (when $a_0 = 0$), where the coefficients (a_0, a_1, a_2, a_3) are homogeneous coordinates of a circle (when $a_0 \neq 0$) or a line (when $a_0 = 0$).

For a set of n points $(X_i, Y_i)(i = 1 \dots n)$ in a plane, we now present a simple and unified method for determining how closely they lie on a circle (or, a line), and then retrieve the homogeneous coordinates (a_0, a_1, a_2, a_3). We first assemble n linear equations of the form 1 into the following matrix form:

$$[B]\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0, \text{ where } [B] = \begin{bmatrix} X_1^2+Y_1^2 & 2X_1 & 2Y_1 & 1 \\ X_2^2+Y_2^2 & 2X_2 & 2Y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ X_n^2+Y_n^2 & 2X_n & 2Y_n & 1 \end{bmatrix} \tag{2}$$

The rank of the matrix $[B]$ determines the solution space of (2). Following Singular Value Decomposition (SVD) method, we first compute the eigenvalues of the 4×4 matrix $[B]^T[B]$. The eigenvector corresponding to the smallest eigenvalue is the solution for (a_0, a_1, a_2, a_3), subject to $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$. The extracted feature is a circle when $a_0 \neq 0$ or a line when $a_0 = 0$.

Furthermore, by looking at the smallest eigenvalue λ_s, one can tell how closely all the points lie on the same feature, whether exactly ($\lambda_s \approx 0$) within the numerical precision or approximately ($\lambda_s \neq 0$). For the latter case of $\lambda_s \neq 0$, this procedure leads to a least-squares solution to (2), which corresponds to a circle with the minimum algebraic fitting error from given points, thus leading to an approximate solution.

In what follows, we show how this data-driven idea for retrieving line or circle feature can be extended to data-driven kinematic synthesis by taking advantage of a planar kinematic mapping.

2.3 Planar Kinematic Mapping

A planar displacement consisting of a translation (d_1, d_2) and a rotation angle ϕ is represented by a planar quaternion $\mathbf{Z} = (Z_1, Z_2, Z_3, Z_4)$ where (see [10, 11] for details),

$$\begin{aligned} Z_1 &= \tfrac{1}{2}(d_1 \sin\tfrac{\phi}{2} - d_2 \cos\tfrac{\phi}{2}), & Z_2 &= \tfrac{1}{2}(d_1 \cos\tfrac{\phi}{2} + d_2 \sin\tfrac{\phi}{2}), \\ Z_3 &= \sin\tfrac{\phi}{2}, & Z_4 &= \cos\tfrac{\phi}{2}. \end{aligned} \tag{3}$$

The components (Z_1, Z_2, Z_3, Z_4) define a point in a projective three-space called the *Image Space* of planar displacements, denoted as Σ. In this way, a 1-DOF (degree of freedom) motion is represented by a curve and a 2-DOF motion is represented by a surface in Σ [11]. The advantage of this formulation is that both the point transformation $[H]$ and line transformation $[\overline{H}]$ associated with a planar displacement are quadratic in Z_i:

$$[H] = \begin{bmatrix} Z_4^2 - Z_3^2 & -2Z_3Z_4 & 2(Z_1Z_3 + Z_2Z_4) \\ 2Z_3Z_4 & Z_4^2 - Z_3^2 & 2(Z_2Z_3 - Z_1Z_4) \\ 0 & 0 & Z_3^2 + Z_4^2 \end{bmatrix}, \tag{4}$$

$$[\overline{H}] = \begin{bmatrix} Z_4^2 - Z_3^2 & -2Z_3Z_4 & 0 \\ 2Z_3Z_4 & Z_4^2 - Z_3^2 & 0 \\ 2(Z_1Z_3 - Z_2Z_4) & 2(Z_2Z_3 + Z_1Z_4) & Z_3^2 + Z_4^2 \end{bmatrix}. \tag{5}$$

where $Z_3^2 + Z_4^2 = 1$. This provides a basis for a unified formulation for including point and line constraints in mechanism synthesis.

2.4 Homogeneous Coordinates of Planar Dyads

As alluded to earlier, both the line and circle constraints can be represented by homogeneous coordinates (a_0, a_1, a_2, a_3). Let (x, y) and (X, Y) denote the coordinates of the moving pivot with respect to M and F, respectively. Then (x, y) and (a_0, a_1, a_2, a_3) are the *design parameters* of the RR and PR dyads. The two dyads are differentiated by whether a_0 takes on zero value.

To obtain unified design equations for both RR and PR dyads, we let $\mathbf{Z}_i = (Z_{i1}, Z_{i2}, Z_{i3}, Z_{i4})$ be the planar quaternion representing the ith position of the end link. Then we have $[X \quad Y \quad 1]^T = [H][x \quad y \quad 1]^T$ where $[\ldots]^T$ denotes column vector and $[H]$ is given by (4). Substituting this into (1), we have shown that the constraints of the end link maps to a quadric surface in Σ [12, 13]:

$$A_{i1}p_1 + A_{i2}p_2 + A_{i3}p_3 + A_{i4}p_4 + A_{i5}p_5 + A_{i6}p_6 + A_{i7}p_7 + A_{i8}p_8 = 0, \tag{6}$$

$$\begin{array}{ll} A_{i1} = Z_{i1}^2 + Z_{i2}^2, & A_{i2} = Z_{i1}Z_{i3} - Z_{i2}Z_{i4}, \\ A_{i3} = Z_{i2}Z_{i3} + Z_{i1}Z_{i4}, & A_{i4} = Z_{i1}Z_{i3} + Z_{i2}Z_{i4}, \\ A_{i5} = Z_{i2}Z_{i3} - Z_{i1}Z_{i4}, & A_{i6} = Z_{i3}Z_{i4}, \\ A_{i7} = Z_{i3}^2 - Z_{i4}^2, & A_{i8} = Z_{i3}^2 + Z_{i4}^2. \end{array} \tag{7}$$

The coefficients p_i are not independent, but must satisfy

$$p_1p_6 + p_2p_5 - p_3p_4 = 0, \quad 2p_1p_7 - p_2p_4 - p_3p_5 = 0. \tag{8}$$

This is because the coefficients p_i are given in terms of (x, y) and (a_0, a_1, a_2, a_3) as:

$$\begin{array}{llll} p_1 = a_0, & p_2 = -a_0x, & p_3 = -a_0y, & p_4 = a_1, \\ p_5 = a_2, & p_6 = -a_1y + a_2x, & p_7 = (a_1x + a_2y)/2, & p_8 = (a_0(x^2 + y^2) + a_3)/4. \end{array} \tag{9}$$

Equations (6) and (8) are said to define the constraint manifolds of RR and PR dyads. As a RR dyad follows a circle constraint, we have $a_0 \neq 0$ and thus $p_1 \neq 0$, whereas a PR dyad follows a line constraint, i.e., $a_0 = 0$ and thus $p_1 = p_2 = p_3 = 0$.

Similarly, using line transformation $[\overline{H}]$, for RP and PP dyads, the constraint manifolds have the same form as (6) and (8) but we have $p_1 = p_4 = p_5 = 0$ for RP dyad and $p_1 = p_2 = p_3 = p_4 = p_5 = 0$ for PP dyad.

Thus, all planar dyads can be represented in the same form as (6) and (8) and by looking at the zeros in the coefficients p_i, we can tell which one of the four dyads that the coefficients $\mathbf{p} = (p_1, p_2, \ldots, p_8)$ correspond to. The set of eight coefficients $\mathbf{p} = (p_1, p_2, \ldots, p_8)$ together with two quadratic relations (8) are said to define *homogeneous coordinates of planar dyads*. This leads to a unified algorithm for simultaneous type and dimensional synthesis of planar dyads. In addition, this formulation eliminates the need for solving a large system of polynomial equations that makes linkage synthesis problem challenging even for dimensional synthesis.

2.5 Constraint Realization via Identification of Planar Dyads

This is the second step of the Data-Driven approach illustrated in the Fig. 1(b). Let $\mathbf{Z}_i$ $(i = 1, 2, \ldots, n)$ denote n planar quaternions associated with n specified task positions of a rigid body. Substituting these planar quaternions into (6) yields n linear equations in p_i. In matrix form they are given by

$$[A]\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_8 \end{bmatrix} = 0, \text{ where } [A] = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} & A_{28} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & A_{n4} & A_{n5} & A_{n6} & A_{n7} & A_{n8} \end{bmatrix} \tag{10}$$

is called the *task matrix* whose elements are given by (7).

As the matrix $[A]^T[A]$ is 8×8 and positive semi-definite, all eigenvalues are non-negative and the eigenvector associated with the smallest of the eight eigenvalues is a "candidate" solution for $\mathbf{p} = (p_1, p_2, \ldots, p_8)$. When $n \leq 5$, the matrix $[A]^T[A]$ has in general $(8 - n)$ identical zero eigenvalues, the null space of $[A]$ is $(8 - n)$ dimensional and is defined by the corresponding orthonormal eigenvectors associated with the zero eigenvalues. Thus, a candidate solution may be expressed as a linear combination of those orthonormal eigenvectors. Imposing (8) on the candidate solution would lead to the homogeneous coordinates of dyads, from which the dyad design parameters such as (x, y) and (a_0, a_1, a_2, a_3) can be obtained by inverting (9).

When $n > 5$, there are, in general, less than three zero eigenvalues. By finding three orthonormal eigenvectors associated with three smallest eigenvalues, one can still obtain a three dimensional eigen-space in which one can impose (8). The result is, of course, an approximate solution to the linear system (10). This means that the spatial arrangement of the given task positions is such that no single DOF motion (or mechanism) can be found that would guide through these positions exactly. In this case, if an approximate solution is not satisfactory, a higher DOF motion (or mechanism) may be required for exact motion generation.

In short, this approach leads to a unified algorithm for both exact synthesis (when $n \leq 5$) and approximate synthesis (when $n > 5$) of planar dyads that can handle joint type and dimensional synthesis simultaneously. By investigating the patterns of zeros in $\mathbf{p} = (p_1, p_2, \ldots, p_8)$, one can also determine which of the four dyads, RR, PR, RP and PP, should be used for the given task [14]. For a set of n task positions, the aforementioned task analysis algorithm may yield up to four dyads from the solution of two quadratic equations in (8), two of which can be combined to form up to six four-bar linkages.

3 MotionGen

We will demonstrate the aforementioned approach for determining type and dimensions of a four-bar linkage via an indigenously developed iOS and Andoid app called MotionGen (Fig. 3) [15]. A critical and early stage goal in the machine design process is generation and evaluation of mechanism design concepts that can potentially drive a machine. The app provides best types and dimensions of four-bar linkages for a given motion parameterized by a set of task positions. The app also provides a touch-based, GUI-driven, feature-rich intuitive environment for simulating existing and synthesized mechanisms, examine branch-, circuit-, and order-defects, import images of existing machines to overlay a new mechanism, export types and dimensions, and reverse-engineer an existing mechanism.

In Synthesis mode, the app is capable of synthesizing linkages for $N(\geq 5)$ task positions for both exact- and approximate- synthesis problems. For a given motion, the designer can select from up to 4 possible dyads of types RR, RP, and PR to assemble a total of up to 6 linkages. The designer can also impose fixed-pivot and fixed-line constraints for less than 5 position problems. Instructions to download and install it on Android platform are available at http://me.eng.sunysb.edu/software. It will be made available soon in the Google Play- and Apple's iTunes-store.

Much of the existing effort in the development of software systems for mechanism synthesis (KINSYN III [16], LINCAGES [17, 18], Sphinxpc [19], Synthetica [20]) focuses on dimensional synthesis. On the other hand, the MotionGen can *compute* both the type and dimensions of the linkages. A detailed review of the state of the art in Computer Aided Mechanism Design can be found in Chase et al. [21].

4 Example

Table 1: Five positions chosen on the hip-joint path of a standard sit-to-stand movement

d_1	d_2	ϕ (degree)
-7.8120	-9.6520	22.90
-1.3590	-6.5340	24.75
0.1180	-2.2530	18.92
1.7690	4.9970	-6.59
5.3100	8.7560	-39.91

We now present a design example for five positions selected carefully on a motion path of the hip joint during sit-to-stand movement. The goal is to design a compact four-bar linkage that interpolates through these positions. In the app, select Input Position icon and enter five positions interactively; Table 1 gives this data. This data can also be accessed from the Examples in the menu under *hip* Motion. As soon as the positions are entered, dyads are computed. Select two dyads (there are only two RR dyads computed in this case). Save dyad parameters data by using *Export Mechanism* function. The pdf file generated can also be emailed.

Table 2: Cartesian Parameters of the Computed Dyads

	Dyad 1	*Dyad 2*
Dyad Type	RR	RR
Fixed Pivot (Global Frame)	(8.8521, 1.8927)	(1.1921,-5.8631)
Moving Pivot (Moving Frame)	(17.4469, -6.4833)	(6.5768, -6.5710)
Length	10.9007	7.2935

Figure 3 shows the final synthesized mechanism and the mechanism parameters are presented in Table 2 – please note that the grid spacing in the figure is scaled with respect to actual units.

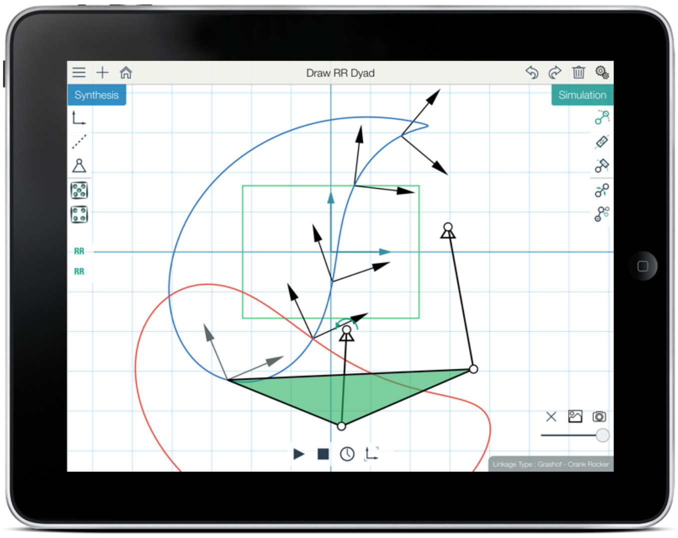


Figure 3: A planar four-bar linkage synthesized for a sit-to-stand motion; the *blue* curve shows the path of the hip joint being traced by the coupler point; the *red*-curve shows the other circuit.

5 Conclusions

In this paper, we have presented 1) unified synthesis equations for the type- and dimensional synthesis of planar linkages for the motion generation problem and 2) an intuitive tool that makes the task of mechanism synthesis completely transparent to designers. In our approach, we do not use the results of dimensional synthesis to evaluate mechanism types enumerated in type synthesis stage. Instead, through task analysis, we seek to extract and identify geometric constraints from the input data about task positions. The identified geometric constraints can then be used to extract both the type and dimensions of a desired mechanism for a given task. The MotionGen has both synthesis and simulation capabilities built in it and provides students and designers a pathway to machine design innovation.

References

[1] L. Tsai, *Mechanism Design: Enumeration of Kinematic Structures According to Function*. CRC Press LLC, 2001.

[2] T. Mruthyunjaya, “Kinematic structure of mechanisms revisited,” *Mechanism and Machine Theory*, vol. 38, pp. 279–320, 2003.

[3] A. G. Erdman and G. N. Sandor, *Mechanism Design: Analysis and Synthesis*, vol. 1. Englewood Cliffs, NJ: Prentice-Hall, 2nd ed., 1991.

[4] K. Sedlaczek, T. Gaugele, and P. Eberhard, *Topology optimized synthesis of planar kinematic rigid body mechanisms*. Multibody Dynamics 2005, ECCOMAS Thematic Conference, 2005.

[5] W. Fang, "Simultaneous type and dimensional synthesis of mechanisms by genetic algorithms," *Mechanism Synthesis and Analysis*, vol. 70, pp. 35–41, 1994.

[6] M. I. Frecker, G. K. Ananthasuresh, S. Nishiwaki, N. Kikuchi, and S. Kota, "Topological synthesis of compliant mechanisms using multi-criteria optimization," *ASME Journal of Mechnaical Design*, vol. 119, no. 2, pp. 238–245, 1997.

[7] A. Saxena and G. K. Ananthasuresh, "A computational approach to the number of synthesis of linkages," *Journal of Mechanical Design*, vol. 125, no. 1, pp. 110–118, 2003. 10.1115/1.1539513.

[8] M. Hayes and P. Zsombor-Murrary, "Towards integrated type and dimensional synthesis of mechanisms for rigid body guidance," in *Proceedings of the CSME Forum 2004*, (London, ON, Canada), pp. 53–61, 2004.

[9] B. Faltings, "Qualitative kinematics in mechanisms," *Artificial Intelligence*, vol. 44, no. 1-2, pp. 89 – 119, 1990.

[10] J. M. McCarthy, *Introduction to Theoretical Kinematics*. Cambridge, MA.: The MIT Press, 1990.

[11] B. Ravani and B. Roth, "Motion synthesis using kinematic mappings," *Journal of Mechanisms Transmissions and Automation in Design-Transactions of the Asme*, vol. 105, no. 3, pp. 460–467, 1983. Article.

[12] Q. J. Ge, P. Zhao, A. Purwar, and X. Li, "A novel approach to algebraic fitting of a pencil of quadrics for planar 4r motion synthesis," *Journal of Computing and Information Science in Engineering*, vol. 12, pp. 041003–041003, 2012.

[13] Q. J. Ge, P. Zhao, and A. Purwar, "A task driven approach to unified synthesis of planar four-bar linkages using algebraic fitting of a pencil of g-manifolds," in *ASME 2013 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, 2013. DETC2013-12977.

[14] Q. J. Ge, P. Zhao, and P. A., *Decomposition of Planar Burmester Problems Using Kinematic Mapping*, vol. 14 of *Advances in Mechanisms, Robotics, and Design Education and Research, Mechanisms and Machine Science*. Springer, 2013.

[15] A. Purwar, Q. J. Ge, and P. Aceves, "Freshman design innovation: Suny innovative instruction technology grant (iitg), $60,000, state university of new york (suny)," 2014.

[16] A. J. Rubel and R. E. Kaufman, "Kinsyn iii: A new human-engineered system for interactive computer-aided design of planar linkages," *ASME Journal of Engineering for Industry*, vol. 99, pp. 440–448, 1977.

[17] A. Erdman and J. Gustafson, "Lincages: Linkage interactive computer analysis and graphically enhanced synthesis packages,," tech. rep., 1977.

[18] A. G. Erdman and D. Riley, "Computer-aided linkage design using the lincages package," in *ASME Design Engineering Technical Conferences*, 1981. 81-DET-121.

[19] D. Ruth and J. McCarthy, "Sphinxpc: An implementation of four position synthesis for planar and spherical 4r linkages," *ASME Design Engineering Technical Conferences*, 1997.

[20] H.-J. Su, C. Collins, and J. McCarthy, "An extensible java applet for spatial linkage synthesis," in *ASME International Design Engineering Technical Conferences*, 2002.

[21] T. Chase, G. Kinzel, and A. Erdman, *Computer Aided Mechanism Synthesis: A Historical Perspective*, vol. 14 of *Advances in Mechanisms, Robotics and Design Education and Research*, pp. 17–33. Springer, 2013.